Sharp charge bounds for a one-scalar chiral U(1) sector

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Abstract

For an isolated chiral $U(1)$ sector with eight Weyl fermions, we minimize the largest fermion charge in absolute value at fixed positive scalar charge h. We require cancellation of the local linear and cubic anomalies and full rank from renormalizable Yukawa masses with one scalar. A reduction from the determinant support to four disjoint pairs yields the sharp bound $(h+7)/2$ for odd $h \geq 9$ and $(h+10)/2$ for even $h \geq 12$. We give explicit minimizers and an independent exhaustive check of the finite exceptions. Bibliographic priority is under review.

1 Domain and statement

Take eight left-handed Weyl fermions that are singlets under the Standard Model. Their charges q_i in an isolated $U(1)$ are nonzero integers of gcd one. Opposite charges are forbidden and repetitions are allowed. We impose $\sum_i q_i = \sum_i q_i^3 = 0$. One complex scalar of charge $h \in \mathbb{Z}_{>0}$ has a nonzero vacuum expectation value. The symmetric mass matrix comes from independent renormalizable Yukawa couplings, with $M_{ij} \neq 0$ allowed only when $q_i + q_j = \pm h$. Let $Q_{\min}(h)$ be the minimum of $\max_i |q_i|$ over the spectra admitting such an invertible matrix.

Theorem 1 *For each positive integer h the minimum exists. Its value is $(h+7)/2$ for odd $h \geq 9$ and $(h+10)/2$ for even $h \geq 12$. The finite table, including overlap checks, is*

h	1	2	3	4	5	6	7	8	9	10	11	12
$Q_{\min}(h)$	8	6	8	7	8	11	11	13	8	13	9	11.

2 Pairing lemma

An invertible matrix has a nonzero term in its determinant. The associated permutation gives eight ordered pairs of fields. Their sums are each h or $-h$ and add to $2\sum_i q_i = 0$, so each sign occurs four times.

Project the pairs to the graph on distinct charges, joining x to $h-x$ or $-h-x$ when those charges are present. After removal of loops the graph is a forest. Indeed, a simple cycle must alternate the two reflection labels since there is at most one neighbor of each label. Each successive pair of reflections around the cycle translates by $2h$ with the same sign, which precludes closure.

At each charge value the incoming and outgoing counts of the permutation agree. Removing a nonloop edge of the forest gives a cut whose directed crossings in the two directions have equal counts. Thus if k fields leave a charge value along an edge, exactly k leave the other endpoint in the reverse direction. Pair those two sets bijectively, doing this for each nonloop edge. All remaining fields come from loops. There is at most one loop value, because the two possible loop charges $h/2$ and $-h/2$ cannot coexist. Its count is even, since the sign of that loop occurs four times in total and every nonloop edge contributes an even number of occurrences. Pair the remaining fields within that charge value. This produces four disjoint pairs of distinct field indices with sums $h, h, -h, -h$. Conversely, those four pairs give an invertible witness by putting ones in both off-diagonal entries of each pair. After reordering, the four two-by-two blocks have determinant -1 each, so the full determinant is 1.

3 Exact minimum

Write $a, b \geq 0$ for the differences in the two pairs of sum h and $c, d \geq 0$ for those of sum $-h$. All four differences have the parity of h. Using

$$u^3 + v^3 = \frac{s^3 + 3s\delta^2}{4}, \qquad s = u + v, \quad \delta = u - v,$$

the cubic equation becomes $a^2+b^2=c^2+d^2$. The two unordered representations must be distinct, since equality would give opposite charges. If they share a difference, equality of the norms forces them to coincide, so their sets of difference values are disjoint. Writing $D = \max(a, b, c, d)$, we have $\max_i |q_i| = (h + D)/2$.

For odd differences, the six unordered pairs of values from $\{1, 3, 5\}$ have distinct squared sums. At $D = 7$, $1^2 + 7^2 = 5^2 + 5^2$ gives equality. For even differences, the fifteen unordered pairs from $\{0, 2, 4, 6, 8\}$ have distinct squared sums. At $D = 10$, $0^2 + 10^2 = 6^2 + 8^2$ gives equality. For $h > D$ each pair lies on the sign of its sum, and opposite charges across signs would require a shared difference. The odd witness is primitive since a pair has difference one. In the even witness the gcd divides two since differences six and eight occur. The positive-sum pairs include $h/2$ and $h/2 + 5$, so at least one charge is odd. The gcd is therefore one. The witnesses attain the two analytic lower bounds in their stated regimes.

For $1 \le h \le 12$, the attached certificates exhibit a witness at each tabulated height, all at most 13. A separate enumeration joins two sorted quartets using their linear and cubic sums as keys. Every sorted eight-charge multiset has exactly one such split. It checks all 168 primitive strictly chiral anomaly-free multisets in $[-13, 13]$. For each h, a row-by-row search for a supported permutation excludes every spectrum below the witness height. Absence of a supported term implies zero determinant for every allowed matrix. The explicit four-block witness supplies the matching upper bound, so the finite exclusions prove the minima.

4 Prior work and scope

Costa, Dobrescu and Fox parametrize the anomaly equations [1]. Berryman et al. examine one-scalar Dirac mass generation and its frequency as a function of the largest fermion charge [2, Section IV]. Bernal et al. classify sectors for effective dimension-five and dimension-six Dirac neutrino mass operators [3, Section III]. The common-center reduction to equal sums of two squares is part of the symmetric Prouhet–Tarry–Escott problem studied by Tsai, Lee and Takahashi [4]. The claim investigated here is the implication from full matrix rank to an off-diagonal pairing in the stated chiral sector, followed by the exact optimum at prescribed h under all of the charge constraints. Priority for that combined result is under review. The model concerns local anomalies and tree-level Yukawa mass matrices.

5 Reproduction

The data and standalone Python verifier are available in the PT-013 report. Use the following command to check the finite minima and mass witnesses with the standard library.

```
python3 verify-pt-013.py.txt
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References

[1] D. B. Costa, B. A. Dobrescu and P. J. Fox, General Solution to the U(1) Anomaly Equations, Phys. Rev. Lett. 123, 151601 (2019).

[2] J. M. Berryman, A. de Gouvêa, D. Hernández and K. J. Kelly, Imperfect Mirrors of the Standard Model, Phys. Rev. D 94, 035009 (2016).

[3] N. Bernal, J. Calle and D. Restrepo, Anomaly-free Abelian gauge symmetries with Dirac scotogenic models, Phys. Rev. D 103, 095032 (2021).

[4] Y.-D. Tsai, J. Lee and F. Takahashi, Arithmetic Symmetry in Ideal Prouhet–Tarry–Escott Solutions, arXiv:2606.07735 (2026).